A Tractable Model of Precautionary Saving in Continuous Time

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This letter derives an analytical solution to the following saving problem. An individual chooses the path of consumption that maximises a time-separable von Neumann-Morgenstern utility function, subject to a standard intertemporal budget constraint. Labour income follows a Poisson process. The consumer is both 'prudent' and 'impatient', and accumulates financial wealth as a buffer against the risk of income loss.

Keywords : optimal consumption; labour income risk; precautionary saving.

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All errors are mine.

[1]I thank John Flemming, *sit tibi terra levitas*. This letter is based on a chapter of my Oxford PhD dissertation (2000). A longer version may be obtained upon request. All errors are mine. The thesis chapter contains an extension that combines saving against the risk of death (Blanchard, 1985) and saving against the risk of retirement, in an economy with many individuals. Since individuals who own different levels of wealth have very different saving behaviour, there is no simple way to aggregate. To keep the model tractable, I assumed that workers can insure each other against the risk of retiring earlier than average, but cannot write insurance contracts with retirees, and so cannot consume their permanent income. I then studied taxation, social security, and dynamic inefficiency.

Introduction

Studies on optimal consumption in risky environments and imperfect markets provide a framework that can match many of the important features of the empirical data on consumption saving and data; see Carroll (2001) and references therein. A difficulty commonly encountered is the mathematical sophistication associated with realistic specifications of the models. We present a tractable model of precautionary saving in continuous time. Labour income is non-diversifiable. Preferences exhibit constant relative risk aversion in consumption (CRRA). Optimal consumption is characterised by concavity with respect to wealth.

The key assumption is that individuals face, throughout their working life, a probability μ of losing their job. Thus their expected working life is $1/\mu$. The parameter μ may be chosen anywhere between zero and infinity. In the limit, as μ goes to zero, individuals face no labour income risk and the precautionary motive disappears. The problem is tractable because once the uncertainty is realized the consumption function may be obtained in closed form. The main advantage of our analytical solution is the use of the CRRA utility and the use of a standard phase diagram analysis. The solution is developed with the familiar tool of optimal control theory in continuous time. Our approach is attractive for modelling the uncertainty associated with rare, large, permanent income losses, such as the risk of serious injury or compulsory retirement. However, it does not capture the more moderate uncertain fluctuations of labour income in the course of an individual's working life. The only source of uncertainty is about the timing of the income loss, and not about its magnitude (this simplification could be relaxed) or about its persistence (this is the key assumption). In this setting, it is possible to give an analytical characterisation of optimal consumption. Individuals engage in buffer-stock saving behaviour, accumulating financial wealth to smooth consumption in the event of an income loss.

1 An Optimal Consumption Problem

Individuals select a flow of consumption C_t to maximise

$$E_t \int_t^{\infty} e^{-\rho(s-t)} \; u(C_s) \; ds$$

where E_t is the conditional expectation operator, and ρ is the rate of pure time preference. The utility index is additively separable in time, and discounted at a constant and positive rate. The utility function is isoelastic,

$$u(C) = \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}},$$

where $\sigma > 0$ is the elasticity of intertemporal substitution. The individual faces a standard budget constraint,

$$\dot{A}_t = r_t A_t + \tilde{Y}_t - C_t,$$

where $\tilde{Y}_t$ is random labour income, A_t is financial wealth, and r_t is the rate of interest. The event that labour income, measured in effective units, drops permanently from w_t to 0 follows a Poisson process with arrival rate μ, where w_t is the wage rate received if working :

$$\tilde{Y}_t = \begin{cases} w_t \; L_t & \text{with probability } \; 1-\mu dt \\ 0 & \text{with probability } \; \mu dt, \end{cases}$$

where L_t denotes the number of effective units. Let L_t grow at a constant (exogenous) rate $\dot{L}_t / L_t = g$.

The deterministic problem faced by the individual whose non-financial income has forever dropped to 0 may be solved independently of the stochastic problem faced by the individual whose labour income is the random variable $\tilde{Y}_t$. It is thus possible to solve the full problem by 'backward induction', solving first for the deterministic problem and secondly for the stochastic problem. Let superscripts index the individual's current state; thus C_t^e stands for consumption when employed, and C_t^u if unemployed.

2 The Euler Equations

2.1 In Permanent Unemployment

The complete solution to this deterministic problem is the standard permanent income result :

$$
\begin{aligned}
C_t^u &= m_t \ A_t \\
(m_t)^{-1} &= \int_t^{\infty} e^{-\int_t^s \ [(1-\sigma)r_v+\sigma\rho] \ dv} \ ds. \\
\dot{A}_t &= r_t A_t - C_t^u.
\end{aligned}
$$

Total wealth is equal to financial wealth A_t. With constant factor prices, $m = (1-\sigma)r + \sigma\rho$.

2.2 With Uncertain Job Tenure

The principle of optimal control yields :

$$
\frac{\dot{C}_t^e}{C_t^e} = \frac{-u'(C_t^e)}{C_t^e \ u''(C_t^e)} \ \left[r_t - \rho - \mu + \mu \ \frac{u'(C_t^u)}{u'(C_t^e)} \right] \tag{1}
$$

Equation (1) is the Euler equation for a worker facing the risk of job loss. The contribution of the precautionary motive to consumption growth depends on the ratio of the marginal utilities between the two states. However, from the assumption that the job loss is expected to be permanent, consumption in unemployment follows a simple permanent income rule, and as a result the marginal utility of consumption $u'(C_t^u)$ is a simple function of financial wealth

A_t. This is the key to tractability. Using $C_t^u = m_t A_t$, and the assumption that the elasticity of intertemporal substitution σ is constant, equation (1) may be written :

$$\dot{C}_t^e / C_t^e = \sigma \ (r_t - \rho + \phi_t)$$
$$\phi_t = \mu \left[\left(\frac{C_t^e}{m_t A_t} \right)^{\frac{1}{\sigma}} - 1 \right]. \tag{2}$$

The Euler equation contains a term, denoted ϕ_t, due to the risk of permanent income loss. It is obvious that $C_t^e > C_t^u$, so that ϕ_t is unambiguously positive. It follows that the precautionary motive *boosts* consumption growth.

3 A Phase Diagram Analysis

Optimal consumption may be characterised by the behaviour of two variables only : consumption while employed, C_t^e, and financial wealth, A_t (from which consumption in unemployment may be deduced, $C_t^u = m_t A_t$). Let c_t denotes consumption per effective units of labour, e.g. $c_t^e \equiv C_t^e / L_t$, and let a_t denote financial wealth per effective units of labour, $a_t \equiv A_t / L_t$. Recall that $\dot{L}_t / L_t = g$. Hence a stationary state in the de-trended variables implies a balanced growth path, or steady state. For a steady state to exist, factor prices r and w must be constant, implying a constant marginal propensity to consume m. Optimal consumption for a worker who faces a risk of permanent job loss is characterised by :

$$\dot{c}_t^e = \sigma \ (r - \rho - \mu - g/\sigma) \ c_t^e + \sigma \ (\mu / m^{1/\sigma}) \ (c_t^e / a_t)^{1/\sigma} \ c_t^e \tag{3a}$$
$$\dot{a}_t = (r - g) \ a_t + w - c_t^e. \tag{3b}$$

Let $m(\rho - r + \mu + g/\sigma)^\sigma > (r - g)\mu^\sigma > 0$. A necessary condition for the existence of a stable saddle-point equilibrium is $\rho > r - \mu - g/\sigma$. This is (a variant of) the 'impatience condition' highlighted by e.g. Merton (1971), Deaton (1991), and Chamberlain and Wilson (2000). A consumer is said to be impatient if, should there be no uncertainty about the future, growth in labour income induces dissaving. For any initial level of financial wealth a_0, there is a unique level of consumption c_0^e such that the consumption function converges to the balanced growth path.

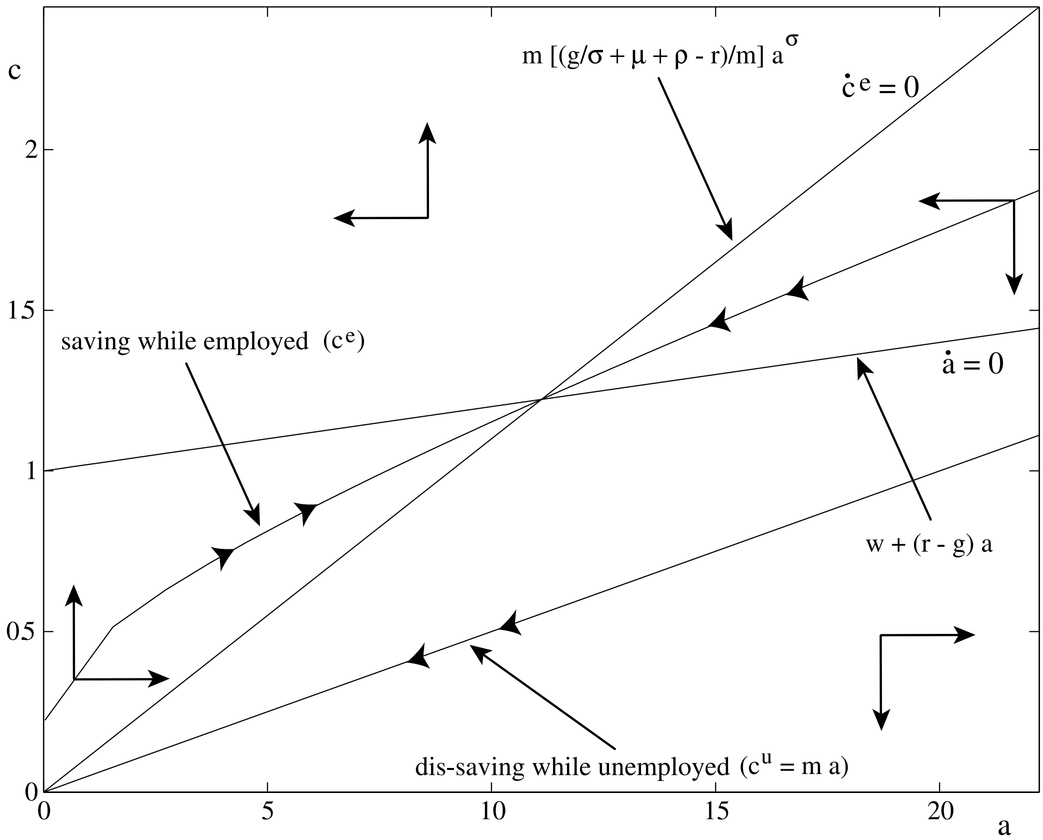

Fig. 1: **Precautionary Saving : Phase Diagram.** With high/moderate elasticity of intertemporal substitution, the consumption function is strongly curved. Calibration : $\sigma = 1$, $\rho = 0.05$, $r = 0.04$, $\mu = 0.025$, $g = 0.02$, $w = 1$. The implied capital income share is about 30%.

Figure 1 shows the phase diagram representing the dynamic adjustment of consumption and financial wealth. The c_t^e locus is the stable manifold that characterises the dynamics of consumption while employed. The consumption function is strictly concave, strongly curved at low levels of financial wealth. Concavity means that the *marginal* propensity to consume is strictly decreasing in wealth, implying that a given change in labour income induces a larger change in consumption at low levels of wealth than at high levels. The c_t^u locus is the consumption function if unemployed ($\rho > r$ implies dis-saving).

There is a 'target' ratio of consumption to financial wealth. The existence of a target ratio was emphasised by Carroll (1992, 1997). The target ratio is the outcome of two conflicting forces: impatience induces dissaving while prudence induces saving. An important implication is that long-run consumption growth is equal to long-run labour income growth, irrespective of whether the rate of interest is constant or tied to the capital stock. As pointed out by Carroll (1992), this explains why consumption tracks income at low frequency but not at high frequency.

The intercept of the consumption function is always strictly positive, ruling out the possibility that the worker might select negative levels of consumption initially in order to accumulate financial wealth fast[2]. This can be understood as follows. From the phase diagram it is clear that the initial consumption level must be positive but no greater than current labour income. The assumption that preferences satisfy $\lim_{c\to 0} u'(c) = \infty$ rules out both consuming nothing and consuming all the income. Consuming nothing would yield negative infinite utility, while consuming even a tiny amount is

[2]This property can be violated if preferences are exponential rather than isoelastic, i.e. if $u(c_t) = -e^{-\eta\ c_t}$, where $\eta > 0$.

feasible and thus clearly better[3] [4]. Consuming all the income means stagnating at the initial zero asset position, and therefore facing the risk of a transition into joblessness with no assets to draw from, which would also yield negative infinite utility[5]. It follows that $c_0^e \in (0, w)$.

For intermediate to high values of the elasticity of intertemporal substitution the consumption function locally displays strong curvature at low levels of wealth, while being approximately linear at higher levels of wealth. It looks as if the consumption function could be well approximated by two linear segments. This contrasts with the numerical simulations of Carroll (1997, 2001) where concavity holds even not very far away from the steady state. The difference between the two findings is in the nature of the uncertainty. In Carroll (1997, 2001) labour income follows an i.i.d. process, whereas here it follows a Poisson jump process. Poisson processes are appropriate for rare stochastic events, such as job loss. However, lower values of the elasticity of intertemporal substitution σ make the consumption profile more smooth,

[3] If the assumption of exogenous market prices were replaced by a general equilibrium assumption, with a neoclassical production function that satisfies the Inada condition at the origin, the consumption function of an economy of many workers would not necessarily have a strictly positive intercept. This can be seen by considering the special case $\mu = 0$ (the standard Ramsey-Solow model). In general equilibrium, the wage is a function of the capital/labour ratio a_t, so that the $\dot{a}_t$ locus goes through the origin. For $a = 0$, any consumption strictly above zero belongs to an unstable region and is therefore not feasible; hence the consumption function goes through the origin. In partial equilibrium, however, provided $w > 0$, the consumption function has a strictly positive intercept. Likewise, in the partial equilibrium Blanchard (1985) model, the consumption function also has a strictly positive intercept.

[4] Note that the positive intercept in this diagram contrasts with a zero intercept for the consumption functions in Deaton (1991) and Carroll (1992). This is because they write consumption as a function of cash-on-hand, including *both* financial wealth and current labour income.

[5] As in Carroll (1997, 2001) the mechanism that rules out borrowing is the risk of a zero-income event. The saving behaviour induced by the presence of this kind of catastrophic risk is similar to that induced by liquidity constraints, as in Deaton (1991). Note that our model can easily deal with the case where labour income drops permanently by a fraction of the original income. This may be done by computing the present value of that part of labour income which is certain and accounting it together with financial wealth A_t. The consumer would never borrow more than can be repaid with certainty.

distributing concavity more evenly across wealth levels.

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