A Tractable Model of Precautionary Saving in Continuous Time

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Last Revised : March 5, 2001

This letter derives an analytical solution to the following saving problem. An individual chooses the path of consumption that maximises a time-separable von Neumann-Morgenstern utility function, subject to a standard intertemporal budget constraint. Labour income follows a Poisson process. The consumer is both 'prudent' and 'impatient', and accumulates financial wealth as a buffer against the risk of income loss.

Keywords : optimal consumption; labour income risk; precautionary saving.

JEL Classification : C61; D11; E24.

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All errors are mine.

[1] This letter is based on a chapter of my Oxford PhD dissertation (2000). I thank John Flemming. A longer version may be obtained upon request. All errors are mine.

Introduction

Studies on optimal consumption in risky environments and imperfect markets provide a framework that can match many of the important features of the empirical data on consumption saving and data; see Caroll (2001) and references therein. A difficulty commonly encountered is the mathematical sophistication associated with realistic specifications of the models. We present a tractable model of precautionary saving in continuous time. Labour income is non-diversifiable. Preferences exhibit constant relative risk aversion in consumption (this could be generalised). Optimal consumption is characterised by concavity with respect to wealth.

The key assumption is that individuals face, throughout their working life, a probability μ of losing their job. Thus their expected working life is $1/\mu$. The parameter μ may be chosen anywhere between zero and infinity. In the limit, as μ goes to zero, individuals face no labour income risk and the precautionary motive disappears. The problem is tractable because the uncertainty is restricted to a single transition. The main advantage of our analytical solution over other existing ones is its greater ability to match empirical facts. It is developed with the familiar tool of optimal control theory and its workings may be illustrated with the help of phase diagrams. Our approach is attractive for modelling the uncertainty associated with rare, large, permanent income losses, such as the risk of serious injury or compulsory retirement. However, it does not capture the more moderate uncertain fluctuations of labour income in the course of an individual's working life. The only source of uncertainty is about the timing of the income loss, and not about its magnitude (this simplification could be relaxed) or about its persistence (this is the key assumption). In this setting, it is possible to give an analytical characterisation of optimal consumption. Individuals engage in buffer-stock saving behaviour, accumulating financial wealth to smooth consumption in the event of an income loss.

1 An Optimal Consumption Problem

Individuals select a flow of consumption C_t to maximise

$$E_t \int_t^\infty e^{-\rho(s-t)} \; u(C_s) \; ds$$

where E_t is the conditional expectation operator, and ρ is the rate of pure time preference. The utility index is additively separable in time, and discounted at a constant and positive rate. The utility function is isoelastic,

$$u(C) = \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}},$$

where $\sigma > 0$ is the elasticity of intertemporal substitution. The individual faces a standard budget constraint,

$$\dot{A}_t = r_t A_t + \tilde{Y}_t - C_t,$$

where $\tilde{Y}_t$ is random labour income, A_t is financial wealth, and r_t is the rate of interest. The event that labour income, measured in effective units, drops permanently from w_t to 0 follows a Poisson process with arrival rate μ, where w_t is the wage rate received if working :

$$\tilde{Y}_t = \begin{cases} w_t \; L_t & \text{with probability } \; 1-\mu dt \\ 0 & \text{with probability } \; \mu dt, \end{cases}$$

where L_t denotes the number of effective units. Let L_t grow at a constant (exogenous) rate $\dot{L}_t / L_t = g$.

The deterministic problem faced by the individual whose non-financial income has forever dropped to 0 may be solved independently of the stochastic problem faced by the individual whose labour income is the random variable $\tilde{Y}_t$. It is thus possible to solve the full problem by 'backward induction', solving first for the deterministic problem and secondly for the stochastic problem. Let superscripts index the individual's current state; thus C_t^e stands for consumption when employed, and C_t^u if unemployed.

2 The Euler Equations

2.1 In Permanent Unemployment

The complete solution to this deterministic problem is the standard permanent income result :

$$
\begin{aligned}
C_t^u &= m_t \; A_t \\
(m_t)^{-1} &= \int_t^{\infty} e^{-\int_t^s \; [(1-\sigma) r_v + \sigma \rho] \; dv} \; ds. \\
\dot{A}_t &= r_t A_t - C_t^u.
\end{aligned}
$$

Total wealth is equal to financial wealth A_t. With constant factor prices, $m = (1-\sigma)r + \sigma\rho$.

2.2 With Uncertain Job Tenure

The principle of optimal control yields :

$$
\frac{\dot{C}_t^e}{C_t^e} = \frac{-u'(C_t^e)}{C_t^e \; u''(C_t^e)} \; \left[r_t - \rho - \mu + \mu \; \frac{u'(C_t^u)}{u'(C_t^e)} \right] \qquad (1)
$$

Equation (1) is the Euler equation for a worker facing the risk of job loss. The contribution of the precautionary motive to consumption growth depends on the ratio of the marginal utilities between the two states. However, from the assumption that the job loss is expected to be permanent, consumption in unemployment follows a simple permanent income rule, and as a result the marginal utility of consumption $u'(C_t^u)$ is a simple function of financial wealth

A_t. This is the key to tractability. Using $C_t^u = m_t A_t$, and the assumption that the elasticity of intertemporal substitution σ is constant, equation (1) may be written :

$$\begin{aligned} \dot{C}_t^e / C_t^e &= \sigma \ (r_t - \rho + \phi_t) \\ \phi_t &= \mu \left[\left(\frac{C_t^e}{m_t A_t} \right)^{\frac{1}{\sigma}} - 1 \right]. \end{aligned} \tag{2}$$

The Euler equation contains a term, denoted ϕ_t, due to the risk of permanent income loss. It is obvious that $C_t^e > C_t^u$, so that ϕ_t is unambiguously positive. It follows that the precautionary motive *boosts* consumption growth.

3 A Phase Diagram Analysis

Optimal consumption may be characterised by the behaviour of two variables only : consumption while employed, C_t^e, and financial wealth, A_t (from which consumption in unemployment may be deduced, $C_t^u = m_t A_t$). Let c_t denotes consumption per effective units of labour, e.g. $c_t^e \equiv C_t^e / L_t$, and let a_t denote financial wealth per effective units of labour, $a_t \equiv A_t / L_t$. Recall that $\dot{L}_t / L_t = g$. Hence a stationary state in the de-trended variables implies a balanced growth path, or steady state. Optimal consumption for a worker who faces a risk of permanent job loss is characterised by :

$$\dot{c}_t^e = \sigma \ (r_t - \rho - \mu - g/\sigma) \ c_t^e + \sigma \ (\mu / m_t^{1/\sigma}) \ (c_t^e / a_t)^{1/\sigma} \ c_t^e \tag{3a}$$

$$\dot{a}_t = (r_t - g) \ a_t + w_t - c_t^e. \tag{3b}$$

Let $m(\rho - r + \mu + g/\sigma)^\sigma > (r - g)\mu^\sigma > 0$. A necessary condition for the existence of a stable saddle-point equilibrium is $\rho > r - \mu - g/\sigma$. This is (a variant of) the 'impatience condition' highlighted by e.g. Merton (1971), Deaton (1991), and Chamberlain and Wilson (2001). A consumer is said to be impatient if, should there be no uncertainty about the future, growth in labour income induces dissaving. For any initial level of financial wealth a_0, there is a unique level of consumption c_0^e such that the consumption function converges to the balanced growth path.

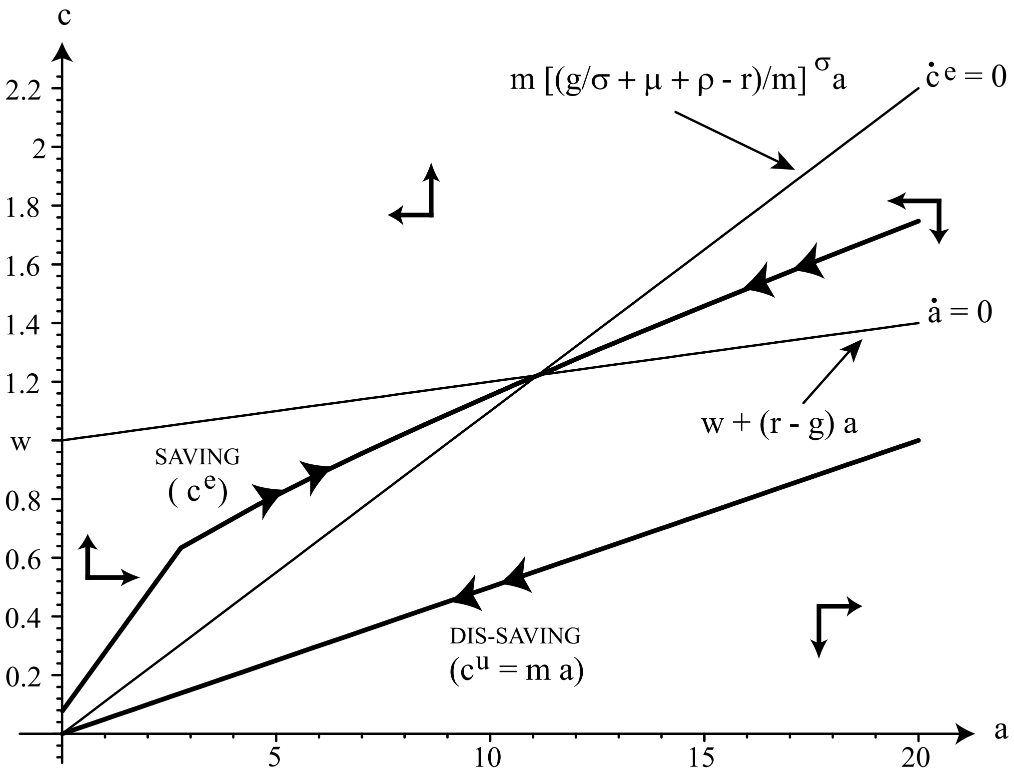

Fig. 1: **Precautionary Saving : Phase Diagram.** With high/moderate elasticity of intertemporal substitution, the consumption function is "kinked". Calibration : $\sigma = 1$, $\rho = 0.05$, $r = 0.04$, $\mu = 0.025$, $g = 0.02$, $w = 1$. The implied capital income share is about 30%.

Figure 1 shows the phase diagram representing the dynamic adjustment of consumption and financial wealth. The c_t^e locus is the stable manifold that characterises the dynamics of consumption while employed. The consumption function is strictly concave, strongly curved at low levels of financial wealth and almost linear at higher levels. Concavity means that the *marginal* propensity to consume is strictly decreasing in wealth, implying that a given change in labour income induces a larger change in consumption at low levels of wealth than at high levels. The c_t^u locus is the consumption function if unemployed ($\rho > r$ implies dis-saving).

There is a 'target' ratio of consumption to financial wealth. The existence of a target ratio was emphasised by Carroll (1997). The target ratio is the outcome of two conflicting forces: impatience induces dissaving while prudence induces saving. An important implication is that long-run consumption growth is equal to long-run labour income growth, irrespective of whether the rate of interest is constant or tied to the capital stock. As pointed out by Carroll (1997), this explains why consumption tracks income at low frequency but not at high frequency. The intercept of the consumption function is always strictly positive, ruling out the possibility that the worker might select negative levels of consumption initially in order to accumulate financial wealth fast. This is a well-known property of optimal consumption paths when preferences satisfy the Inada condition $\lim_{c\to 0} u'(c) = \infty$[2]. An implication is that the growth rate of consumption goes to infinity as the level of financial wealth goes to zero.

For intermediate to high values of the elasticity of intertemporal substitution the consumption function locally displays strong curvature, resembling a kink, while being approximately linear beyond a certain level of wealth. It looks as if the consumption function could be well approximated by two linear segments. This contrasts with the numerical simulations of Carroll (1997) where concavity holds even not very far away from the steady state. The difference between the two findings is in the nature of the uncertainty. In Carroll (1997) labour income follows an i.i.d. process, whereas here it follows a Poisson jump process. Poisson processes are appropriate for rare stochastic events, such as job loss. However, lower values of the elasticity of intertemporal substitution σ make the consumption profile more smooth, distributing concavity more evenly across wealth levels.

[2]This property can be violated if preferences are exponential rather than isoelastic, i.e. if $u(c_t) = -e^{-\eta\ c_t}$, where $\eta > 0$.

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Extensions

Extension 1: Discussion

There is substantial evidence that precautionary saving can be large, especially when safe financial wealth is small relative to risky human wealth. To understand the impact of uncertainty on saving, it is therefore crucial to go beyond certainty equivalence, see e.g. Skinner (1988), Zeldes (1989), Deaton (1991), Carroll (1997), Attanasio, Banks, Meghir, and Weber (1999), Gourinchas and Parker (2001).

A difficulty commonly encountered is the mathematical sophistication associated with realistic specifications of the models, in particular with preferences, stochastic processes, and the nature of markets. One approach is to approximate the model solution numerically and to perform calibration experiments to learn about the role of the structural parameters. Another approach is to make assumptions that allow the problem to be solved analytically rather than by numerical simulations. Explicit solutions make it possible to compute the exact response of endogenous variables to exogenous shocks, and to extract restrictions that can be tested empirically. To date, unfortunately, explicit solutions are known in only a few cases of little empirical relevance. Examples : (i) Merton (1971) and Karatzas, Lehoczky, Sethi and Shreve (1986) allow fairly general preferences and stochastic processes but assume diversifiable labour income; (ii) Flemming (1978) and Kimball and Mankiw (1989) allow fairly general stochastic processes and non-diversifiable labour income but assume preferences that exhibit constant absolute risk aversion in consumption (exponential utility). While much can be learned from these studies, each leaves something to be desired. For instance, if labour income is fully diversifiable, optimal consumption is proportional to total accounting wealth; while if preferences are exponential, optimal consumption is linear in total accounting wealth. Neither of these implications is satisfactory. Carroll and Kimball (1996), for one, show that the consumption function is typically concave in total wealth. Consumers with low wealth should have higher expected consumption growth. This piece of information is crucial, say, to gauge the impact of a tax cut on aggregate consumption. It is therefore of practical as well as theoretical interest to develop tractable models of precautionary saving that allow for non-diversifiable labour income as well as more reasonable specifications of risk aversion; e.g. decreasing absolute risk aversion.

Extension 2: Transitional Dynamics

Comparative Statics and Dynamics[3]
What is the effect of a rise in the probability of unemployment ? A rise in μ implies a higher boost ϕ_t to consumption growth. The consumption function is thus shifted downwards and its slope is reduced. However, the worker's target level of financial wealth rises. Overall, a rise in μ raises steady-state consumption. Starting *below* the stationary state, consumption immediately falls upon realisation of the shock, and gradually rises to its new long-run level as financial wealth is accumulated. Consumption undershoots its long-run level. The other effects are as follows. A rise in g lowers the target level of financial wealth and shifts the adjustment path upwards. A rise in ρ lowers the target level of financial wealth and shifts the adjustment path upwards. A rise in r raises the target level of financial wealth and shifts the adjustment path downwards.

Speed of Wealth Accumulation
How fast does financial wealth accumulate ? Given the simplicity of the model, it is possible to compute the speed of convergence of the system. The

[3]All shocks to parameters are unexpected and permanent.

negative eigenvalue associated with the stable manifold is equal to :

$$\frac{\mathrm{Tr}}{2}\left(1-\sqrt{1-\frac{4\mathrm{Det}}{\mathrm{Tr}^2}}\right).$$

where $\mathrm{Tr} = \rho+\mu+(1-\sigma)g/\sigma$ and $\mathrm{Det} = (\rho+\mu+g/\sigma-r)\,[(r-g)-(\sigma\rho+(1-\sigma)r)\mu^{-\sigma}(\rho+\mu+g/\sigma-r)^{\sigma}]$. The higher the probability of unemployment μ, the slower financial wealth is accumulated. The other effects are as follows. A higher rate of growth of labour income g raises the speed. A higher level of the real interest rate r reduces the speed. A higher rate of impatience ρ raises the speed. A higher elasticity of intertemporal substitution σ (lower coefficient of instantaneous relative risk aversion $1/\sigma$) raises the speed.

Extension 3: Cash-On-Hand Analysis

Equations (3a)-(3b) is a characterisation of the dynamics in terms of consumption and financial wealth, as in standard Graduate textbook treatments of the Ramsey model. Another way of looking at the same results is in terms of expected consumption growth and cash on hand, as in e.g Deaton (1991) and Carroll (1997). A simple change of variables achieves this. Let x_t denote cash on hand, per effective units of labour, when employed, $x_t = ra_t + w$, and let y_t denote expected consumption growth, $y_t = \dot{C}^e_t/C^e_t$.

$$\begin{aligned}\dot{x}_t =& gw + (r-g)x_t - m\ \left[\frac{y_t+\sigma(\rho-r)+\sigma\mu}{\sigma\mu}\right]^{\sigma}\ (x_t - w)\\ \dot{y}_t =& \frac{1}{\sigma}(y_t+\sigma(\rho-r)+\sigma\mu)\ \left(y_t + m\ \left[\frac{y_t+\sigma(\rho-r)+\sigma\mu}{\sigma\mu}\right]^{\sigma} - \frac{rx_t}{x_t-w}\right)\end{aligned}$$

Figure 2 depicts expected consumption growth y_t as a function of cash on hand x_t. Figure 2 should be compared with Figure Ia in Carroll (1997). The horizontal line at $\sigma(r-\rho)$ shows the growth rate of consumption when the uncertainty vanishes $\mu \to 0$ (the standard Ramsey model). The horizontal line at g shows the growth rate of (effective units of) labour income. The vertical line at x^* shows the target value of cash-on-hand (per effective units

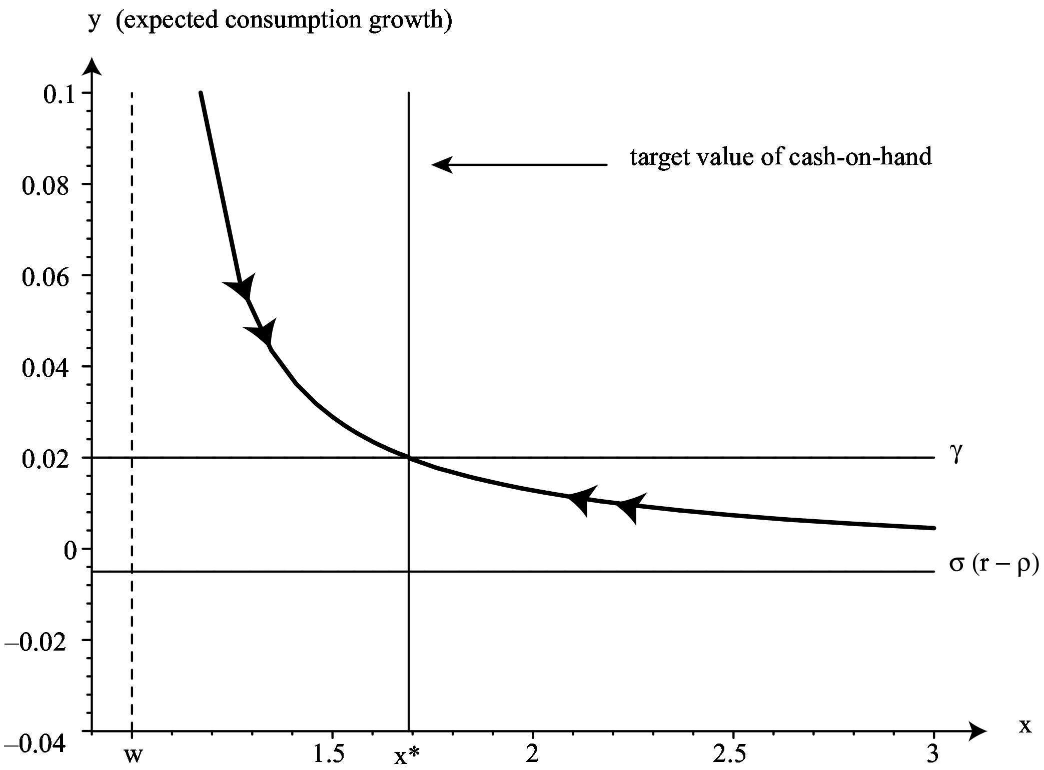

Fig. 2: **Expected Consumption Growth as a Function of Cash On Hand.** Calibration : $\sigma = 1/2$ (intermediate intertemporal substitution / prudence), $\rho = 0.05$, $r = 0.04$, $\mu = 0.025$, $g = 0.02$, $w = 1$.

of labour income). The intersection of the x^* line and g line is the balanced growth equilibrium. If the individual is sufficiently impatient, the target is stable in the saddle-point sense, a property illustrated by arrows of motion along the path of expected consumption growth[4]. As $x_t \rightarrow w$, $y_t \rightarrow \infty$. Indeed, as cash on hand x_t shrinks to the value of current income w, financial wealth a_t goes to 0, and the marginal utility of consumption goes to infinity. As $x_t \rightarrow \infty$, $y_t \rightarrow \sigma(r-\rho)$. Indeed, as financial wealth grows without bound more consumption is financed by safe financial wealth rather than by risky labour income, and the importance of uncertainty vanishes. If the individual is sufficiently impatient, the limit as $x_t \rightarrow \infty$ is irrelevant since it is never optimal to hold more cash on hand than the target level. By contrast, in the absence of a bequest motive, the individual enters the labour market without any financial wealth and, therefore, the limit as $x_t \rightarrow 0$ is highly relevant. *If the worker is sufficiently impatient, he has a target ratio of cash-on-hand to permanent-income.*

Extension 4: More References

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[4] Saddle-point stability follows from the preceding analysis since $(x_t,\, y_t)$ is merely a transformation of $(c_t^e,\, a_t)$.

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Extension 5: Appendix

Formally, the problem may be written as a pair of partial differential equations, one for each possible state[5]

$$\max_{C^e} \left\{ u(C_t^e) + \frac{\partial V_t^e}{\partial A_t} \left(r_t A_t + w_t L_t - C_t^e \right) \right\} + \mu \left(V_t^u - V_t^e \right) + \frac{\partial V_t^e}{\partial t} - \rho V_t^e = 0 \tag{4a}$$

$$\max_{C^u} \left\{ u(C_t^u) + \frac{\partial V_t^u}{\partial A_t} \left(r_t A_t - C_t^u \right) \right\} + \frac{\partial V_t^u}{\partial t} - \rho V_t^u = 0. \tag{4b}$$

The arguments of the value functions have been omitted for clarity; $V^e(A_t, t)$ denotes the value function of the individual in work, and $V^u(A_t, t)$ denotes the value function of the individual in unemployment (short-hand notation V_t^e and V_t^u).

Equation (4a) is the standard Hamilton-Jacobi-Bellman (HJB) equation for the optimal consumption problem under uncertainty. Equation (4b) is the standard optimality condition of the deterministic problem. The only difference between the two equations is the presence of $\mu(V_t^u - V_t^e)$ in (4a), the expected reduction in the value function due to the income loss. The key to tractability is that the deterministic problem in (4b) may be solved

[5] A good introduction to continuous-time stochastic dynamic optimization is Dixit and Pindyck (1994), chapter 4.

independently of the stochastic problem in (4a). The solution of equation (4b) may be inserted into equation (4a), which may then be solved.

The HJB equation in (4a) may be used to derive an Euler equation that is necessary for an optimal path. Maximisation of the right-hand-side of (4a) yields the envelope relation :

$$u'(C_t^e) = \frac{\partial V^e(A_t, t)}{\partial A_t}. \tag{5}$$

Given sufficient time, the value function converges to a stationary form, implying $\frac{\partial V_t^e}{\partial t} = 0$. Differentiating (4a) with respect to A_t, substituting the envelope relation (5), and dropping the time argument in the value function,

$$(r_t\ A_t + w_t\ L_t - C_t^e)\ \frac{\partial^2 V^e(A_t)}{\partial A_t^2} + (r_t - \rho - \mu)\ \frac{\partial V^e(A_t)}{\partial A_t} + \mu\ \frac{\partial V^u(A_t)}{\partial A_t} = 0. \tag{6}$$

Differentiating the envelope relation (5) with respect to time yields

$$u''(C_t^e)\ \dot{C}_t^e = \frac{\partial^2 V^e(A_t)}{\partial A_t^2}. \tag{7}$$

Substituting equations (5) and (7) into (6) yields equation (1) above.