Multiple BGPs in a Growth Model with Habit Persistence : A Comment : Long Version

Patrick Toche

Abstract

Recently in this journal, Chen (2007) analyzes a model of an economy without distortions in which rational forward-looking households care about how their current consumption compares with their own past consumption. Chen claims the existence of two saddle-point stable balanced growth paths. In this comment, I show that the claim is incorrect. There is a unique saddle-point stable balanced growth path and no global indeterminacy.

Key words: habit formation, relative consumption, growth
JEL Classification: C62, D91, E21

* Address for Correspondence : P. Toche, Department of Economics and Finance, City University of Hong Kong, Tat Chee Avenue, Kowloon Tong, Hong Kong.
Email address: ptoche@cityu.edu.hk (Patrick Toche).

[1] I thank the Editor for his diligence.

Submitted to : J. Money, Credit, Banking. *Last revised : 7 August 2008*

1 Introduction

Recently in this journal, Chen (2007) analyzes a model of an economy without distortions in which rational forward-looking households care about how their current consumption compares with their own past consumption. The habit formation mechanism considered by Chen is a generalization of the mechanism introduced by Ryder and Heal (1973) and studied by Carroll, Overland, and Weil (1997, 2000) in a growing environment; and a special case of the mechanism considered by Iannaccone (1986) in a stationary environment. Chen claims to prove the existence of two saddle-point stable balanced growth paths. In this comment, I show that his claim is incorrect : there is a unique saddle-point stable balanced growth path and no global indeterminacy.

I summarize Chen's model, correct his propositions, and offer a simple proof that the balanced growth path is unique.

2 Chen's Model

2.1 The Optimization Problem

I use Chen's notation throughout. I also find it convenient to introduce the following new notation. Let $r = A - \delta_k > 0$ denote the rate of interest. Let $g(c, S)$ stand for the input in the habit formation mechanism, and let $U(c, S)$ denote the utility function, both defined explicitly below.

The problem is to select a consumption path to maximize an intertemporal utility function defined in terms of consumption and a reference habit stock,

$$\max_{c} \int_{0}^{\infty} e^{-\rho t} \, U(c_t, S_t) dt \tag{1}$$

subject to a budget constraint :

$$\dot{k}_t = r k_t - c_t. \tag{2a}$$

and a habit formation mechanism :

$$\dot{S}_t = g(c_t, S_t) - \delta_S S_t. \tag{2b}$$

The optimal control program consists in maximizing the objective function (1) subject to the dynamic constraints (2a) and (2b), where c_t is a control variable, and S_t and k_t are two state variables – c_t is the flow of consumption, k_t is the capital stock, with $k_0 > 0$ given, S_t is the reference index to which consumption is compared, with $S_0 > 0$ given, ρ is the discount rate, and δ_S is the depreciation rate of the habit stock S_t.

2.2 The Assumptions

Assumption (Harmful Habits)**.**
The utility function is $U(c, S) = u(c/S^\gamma) = (1-\sigma)^{-1}(c/S^\gamma)^{1-\sigma}$*, where* $\sigma > 1$ *and* $0 < \gamma < 1$*. The input in the habit formation mechanism is* $g(c, S) = Bc^\mu S^{1-\mu}$*,* $0 < \mu \leq 1$ *and* $B > 0$*. The stock of habit depreciates at rate* $\delta_s > 0$*. These functional forms and parameter restrictions imply that the optimizing agent experiences "harmful" habits – in Iannaccone (1986)'s terminology –, in the sense that* $U_c(c, S) > 0$*,* $U_S(c, S) < 0$*, and* $g_c(c, S) > 0$*. Habit formation is internal to the agent who takes into account the effect of current consumption on future habits – that is, the effect of* c *on* $g(c, S)$*. The special case* $\mu = 1$ *reduces to the model Carroll et al. (1997) analyze.*

2.3 The Optimality Conditions

The optimality conditions are correctly stated in Chen (2007), his equations (5a), (5b), (5c), (5d) [2]. I re-state the conditions here because I will use them in section 3.4 to obtain a direct proof of uniqueness of the balanced growth path. To air the notation, I occasionally drop the time index and the arguments of $U(c, S)$ and $g(c, S)$.

The current-value Hamiltonian is :

$$H = U(c, S) + \lambda_k(rk - c) + \lambda_S(\delta_S S - g(c, S))$$

where λ_k is the shadow price of the capital stock, and $-\lambda_S$ is the shadow price of the habit stock. Note that Chen adopts the notational convention of positive multipliers. Since the model assumes harmful habits (see the assumption above), it follows that λ_S measures the absolute value of the price of the (harmful) habit stock (a negative price). We must be careful, when deriving the Hamiltonian conditions, since λ_S is associated with $-\dot{S}$. The necessary and sufficient conditions for an optimum are the stock constraints (2a) and

[2] Chen's equation (5a) contains a typo. The minus sign in front of λ_S should be a plus sign.

(2b), the first-order conditions (3a)-(3c) :

$$\frac{\partial H}{\partial c} = U_c - \lambda_k - \lambda_S g_c = 0 \tag{3a}$$

$$\dot{\lambda}_k = -\frac{\partial H}{\partial k} + \rho\lambda_k = (\rho - r)\lambda_k \tag{3b}$$

$$\dot{\lambda}_S = +\frac{\partial H}{\partial S} + \rho\lambda_S = (\rho + \delta_S - g_S)\lambda_S - U_S \tag{3c}$$

together with the transversality conditions (3d)-(3e) :

$$\lambda_k(t) \geq 0 \qquad \lim_{t\to\infty} e^{-\rho t}\lambda_k(t)k(t) = 0 \tag{3d}$$

$$\lambda_S(t) \geq 0 \qquad \lim_{t\to\infty} e^{-\rho t}\lambda_S(t)S(t) = 0 \tag{3e}$$

3 The Balanced Growth Path

3.1 Definition

Definition (Balanced Growth Path)**.**
A balanced growth path is a solution to the dynamic optimization problem such that the growth rates of consumption $\dot{c}_t/c_t$, capital $\dot{k}_t/k_t$, and habit $\dot{S}_t/S_t$ are constant (not necessarily equal, possibly zero).

3.2 The Corrected Propositions

I re-state Proposition 2 first as Propositions 3 and 1 follow from it.

Proposition 2 (Revised)**.**
In this economy with habit formation, under some restrictions on the parameters, there exist a unique balanced growth path characterized by saddle-point equilibrium dynamics. Given initial values of capital $k_0 > 0$ and habit $S_0 > 0$, there is a unique value of consumption $c_0 > 0$ that converges to the saddle point (and satisfies the transversality conditions).

Proposition 3 (Revised)**.**
There is no global indeterminacy of the balanced growth path.

Proposition 1 (Revised)**.**
The long-run elasticity of substitution, defined as the derivative of long-run consumption growth $\dot{c}_t/c_t$ with respect to the interest rate $r = A - \delta$, is equal to $[\gamma(1-\sigma)+\sigma]^{-1}$. The elasticity is strictly greater in an economy with harmful

habit formation if $\sigma > 1$ (the conditions for harmful habits are given above). The elasticity is independent of the habit formation parameters μ, B, and δ_s (these parameters must be positive and non-zero).

Propositions 2 and 3 state that the balanced growth path of Chen's model is the same as Carroll et al. (1997, 2000). Proposition 1 states that the introduction of the parameter μ in the habit formation mechanism has no effect on the long-run elasticity of substitution. The non-linearity implied by $\mu > 0$ does not affect the long-run equilibrium. Proofs and further discussion follow.

3.3 Proofs and Discussion of the Corrected Propositions

All the results stated above are extensions of Carroll et al. (1997). It is straightforward to adapt their method of analysis to the case $\mu < 1$. To keep this comment short, I therefore refer the reader to Carroll et al. (1997) for a complete analysis of stability that can be adapted to Chen's model. In this section, I discuss the step where Chen's analysis is incorrect and then, in section 3.4, outline a simple proof that the balanced growth path is unique. This proof is of independent interest because it addresses the problem from a somewhat different angle from either Carroll et al. (1997) or Chen (2007).

Both Carroll et al. (1997) and Chen (2007) derive optimality conditions from the Hamiltonian approach to optimal control. The Hamiltonian conditions can be simplified because in balanced growth some variables are proportional to each other. The purpose of simplification is to transform the dynamic system formed by the growing variables c_t, k_t, S_t into one formed by some stationary variables. Simplifications can be done in several ways. As it turns out, some simplifications are easier to deal with than others. Carroll et al. (1997) use the transformation $\dot{c}/c$, c/S and k/S, while Chen (2007) uses the transformation $\lambda = \lambda_s/\lambda_k$, $x = c/S$, $z = k/S$. Chen's choice of transformation yields complicated equations, and is prone to error.

Because the capital stock k affects neither preferences $U(c, S)$ nor habit formation $g(c, S)$, the dynamic system is recursive and local stability does not depend on k. We can therefore limit the dynamic analysis to a sub-system such as $(\dot{c}/c, x)$ or (λ, x).

The analysis of Carroll et al. (1997) offers some insights about what could go wrong when using a transformation such as $\lambda = \lambda_s/\lambda_k$. They uncover two candidate balanced growth paths and show that one of them violates the transversality condition (3e). They show that along this candidate path the shadow price of capital is zero, $\lambda_k = 0$. Chen's analysis goes wrong when he studies the asymptote $\lambda \to \infty$. He misses the fact that when λ_k crosses zero it changes sign, and therefore so does λ. As a consequence, in *his* Figure 1, he incorrectly locates one of the branches associated with $\dot{x} = 0$. He places both branches in the positive quadrant, whereas one of the branches actually lies in the negative quadrant.

This can be shown explicitly using Chen's equations. Chen's (λ, x, z) dynamic system, represented by *his* equations (7a)-(7c), is correctly written. It is the analysis of equation (7a) which yields the incorrect branch of $\dot{x} = 0$. It is possible to compute the $\dot{x} = 0$ locus in closed form. First, his equation (7a) can be re-written (for $\lambda \neq 0$, $x \neq 0$) :

$$\begin{aligned} [\sigma + (\sigma + \mu - 1)\lambda g_c] \;\; \dot{x}/x = r - \rho - \lambda g_c \, [\delta_S + \rho - \gamma(x/\lambda) - \Omega B x^{\mu}] \\ - (1 + \lambda g_c)\Delta(B x^{\mu} - \delta_S) \end{aligned} \tag{7a$'$}$$

where $\Omega = 1 - \mu + \mu\gamma > 0$, and $\Delta = \sigma + (1 - \sigma)\gamma > 0$. Δ is equal to the inverse of the long-run elasticity of substitution. His equation (7b) is re-written here for future reference (for $\lambda \neq 0$) :

$$\dot{\lambda}/\lambda = r + \delta_S - \gamma(x/\lambda) - \Omega B x^{\mu} \tag{7b$'$}$$

The right-hand side of equation (7a$'$) is non-linear in x but linear in λ. Consequently, rather than analyze the system in the (λ, x) plane, it is easier to do it in the (x, λ) plane.

Figure 1 depicts the phase diagram associated with the sub-system (x, λ). The thick line dressed with arrows is the stable arm of the saddle-point system. An intersection of the $\dot{\lambda} = 0$ line with the $\dot{x} = 0$ line gives a stationary point of the sub-system (x, λ), which is then a candidate balanced growth path of the system (c, k, S). Depending on parameter values, there are one or two candidate stationary points : a point in the strictly positive quadrant and, possibly, the origin[3]. There are no parameter values for which $\dot{x} = 0$ and $\dot{\lambda} = 0$ cross twice in the strictly positive quadrant[4]. The small arrows of motion indicate that the origin is globally unstable, whereas the other stationary point is saddle-point stable. Given an initial value of $x_0 = c_0/S_0$ in the positive

[3] In Figure 1, the branch of $\dot{x} = 0$ located to the left of $\hat{x}$ touches the origin. If $\mu = 1$, this branch lies strictly below the origin – ruling out the origin even as a candidate stationary point. There are no parameter values for which this branch lies strictly above.

[4] See my direct proof that the balanced growth path is unique in section 3.4. The $\dot{\lambda} = 0$ locus is strictly increasing. The $\dot{x} = 0$ locus can be non-monotonic, for low values of μ, but is strictly decreasing in a neighborhood of $\hat{x}$ and in a neighborhood of $\tilde{x}$. Since the $\dot{\lambda} = 0$ locus is strictly increasing, if they cross twice they must cross thrice within $(\hat{x}, \tilde{x})$, including two stable configurations. A formal proof that $\dot{x} = 0$ and $\dot{\lambda} = 0$ cross only once appears to be tedious, illustrating the usefulness of my direct proof.

quadrant, there is a unique value of $\lambda_0 = \lambda_{s,0}/\lambda_{k,0}$ that places the agent on the stable arm that leads to the stationary state[5].

The locus associated with $\dot{x} = 0$ is given explicitly by :

$$\dot{x} = 0 \Rightarrow \ \lambda = \frac{(r - \rho + \Delta\delta_s)x^{1-\mu} - (\Delta - \gamma\mu)Bx}{B\left[\rho - \delta_s(\Delta - 1) + (\Delta - \Omega)Bx^{\mu}\right]}$$

where $\Delta > 1 > \gamma\mu$, and $\Delta - \Omega = (1-\gamma)(\sigma + \mu - 1) > 0$. Chen correctly identifies an asymptote, where the denominator of $\dot{x} = 0$ is annulled. The unique value of x which defines the asymptote, $\hat{x}$, may be written in terms of the model parameters :

$$\hat{x} = \left[\frac{(\sigma - 1)(1-\gamma)\delta_S - \rho}{B(1-\gamma)(\sigma + \mu - 1)}\right]^{1/\mu}$$

for $\rho \leq (\sigma - 1)(1-\gamma)\delta_S$ – the asymptote lies in the negative half-plane otherwise. The unique value of x such that the numerator of $\dot{x} = 0$ is annulled, $\tilde{x}$, may be written :

$$\tilde{x} = \left[\frac{r - \rho + [\sigma + (1-\sigma)\gamma]\delta_S}{B[\sigma + (1-\sigma)\gamma - \gamma\mu]}\right]^{1/\mu}$$

for $\rho < r$ – the agent cannot sustain positive growth otherwise. Any point on the $\dot{x} = 0$ locus that lies strictly to the left of $\hat{x}$ is associated with a negative shadow price of capital, $\lambda_k < 0$. Any point on the $\dot{x} = 0$ locus that lies strictly to the right of $\tilde{x}$ is associated with a positive shadow price of harmful habits, $\lambda_S > 0$. A few manipulations yield that $\hat{x} < \tilde{x}$, under our parameter restrictions $\gamma < 1$, $\mu < 1$, $\sigma > 1$, $r > \rho$. An optimum stationary state of the (x, λ) system must therefore lie within $(\hat{x}, \tilde{x})$. This proves that the origin cannot be an

[5] I omit a proof of saddle-point stability of the stationary state. My comment is centered on the issue of uniqueness of the equilibrium. A proof of stability can be adapted from Carroll et al. (1997). The numerical simulation of Figure 1 illustrates the saddle-point stability of the balanced growth path with $\mu = 1/2 < 1$.

optimum balanced growth path of (c, k, S)[6] [7].

The locus associated with $\dot{\lambda} = 0$ is given explicitly by :

$$\dot{\lambda} = 0 \Rightarrow \ \lambda = \frac{\gamma x}{r + \delta_s - \Omega B x^{\mu}}$$

Differentiation immediately yields that the $\dot{\lambda} = 0$ locus is strictly increasing for $\lambda > 0$ and $x > 0$. The $\dot{\lambda} = 0$ locus goes through the origin and has an asymptote at the unique value of x such that the denominator is annulled[8].

To sum up, Propositions 2 and 3 can be established from an analysis of equations (7a′) and (7b′). Chen incorrectly locates one of the branches associated with the $\dot{x} = 0$ locus. The mistake appears explicitly in *his* equation (8). Consequently, the stationary state labeled L in *his* Figure 1 does not in fact exist. There is a unique balanced growth path[9].

Proposition 1 requires further discussion. Chen misinterprets *his* equation (6b)

---

[6] The origin violates the transversality condition (3e), which can be written in growth rates as $\dot{\lambda}_S/\lambda_S + \dot{S}/S - \rho < 0$. Following Carroll et al. (1997), since $\lambda_k < 0$, we have :

$$\frac{\dot{\lambda}_S}{\lambda_S} + \frac{\dot{S}}{S} - \rho = r - \rho + (1-\gamma)\mu B x^{\mu} - \gamma x/\lambda > 0.$$

The inequality follows from $r > \rho$, $\gamma < 1$, and $\lambda = \lambda_S/\lambda_k \leq 0$.

[7] In *his* footnote 13, Chen (2007) identifies the origin as a candidate optimum stationary state and correctly rules it out. Chen states that the origin has "zero consumption and shadow price of habits." In addition, *crucially*, the origin has a strictly negative shadow price of capital if $\rho < (\sigma - 1)(1 - \gamma)\delta_S$ – the condition implying $\hat{x} > 0$ and labeled condition R by Chen.

[8] The $\dot{\lambda} = 0$ locus is negative and increasing for all values of x that lie to the right of the asymptote in question.

[9] There are two mistakes in *his* equation (8). One branch of the $\dot{x} = 0$ locus is incorrectly located on the positive quadrant. The other branch is incorrectly stated to be monotonic, whereas it can be non-monotonic for some values of the parameters. It is the first mistake that leads Chen to claim the existence of one balanced growth path to the left of $\hat{x}$, while the second mistake leads him to ignore the possibility of two balanced growth paths to the right of $\hat{x}$. I can rule out the existence of two balanced growth paths to the right of $\hat{x}$ based on numerical simulations of the (x, λ) system and based on a direct proof not involving x and λ. See my footnote 4 and section 3.4.

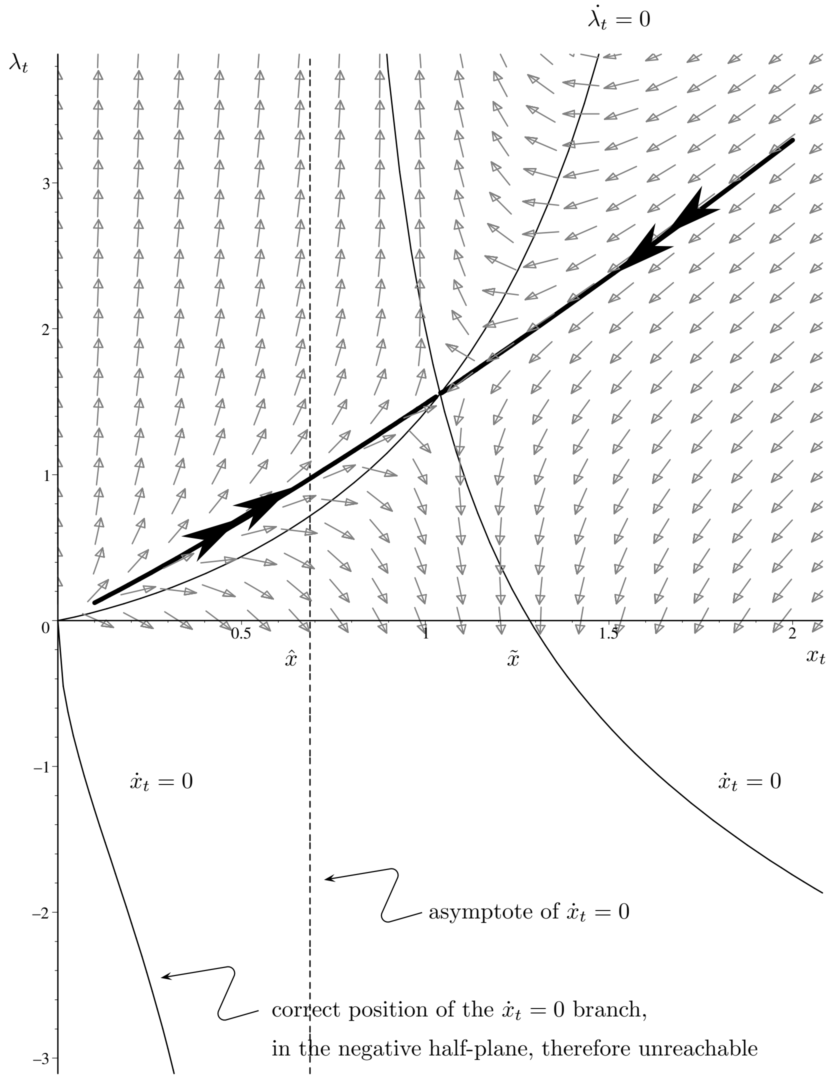

Fig. 1. **The Unique Balanced Growth Path** The phase diagram of the transformed system (x, λ) shows the two branches of $\dot{x} = 0$ separated by an asymptote at $x = \hat{x} \simeq 0.7$. The branch located on the left of $\hat{x}$ lies in the negative quadrant for $x \geq 0$, with the exception of a contact with the origin (this branch is associated with a negative shadow price of capital, $\lambda_k < 0$). The phase diagram also shows that the origin is a globally unstable stationary state – the transversality condition (3b) is violated. The stationary state must therefore lie to the right of $\hat{x}$. The branch located on the right of $\hat{x}$ crosses zero at $x = \tilde{x} \simeq 1.3$. The portion of the branch located on the right of $\tilde{x}$ lies in the negative quadrant (this branch is associated with a positive shadow price of (harmful) habit, $-\lambda_S > 0$). An optimum stationary state must lie between $\hat{x}$ and $\tilde{x}$, and is unique. Arrows of motion show the saddle–point stability of the positive stationary state. Benchmark simulation : $\rho = 0.05$, $\gamma = 0.5$, $\sigma = 4$, $\mu = 0.5$, $\delta_S = 1$, $r = A - \delta_k = 0.1$, $B = 1$. The stationary state is approximately (1.0, 1.6). Arrows are not shown for negative values of λ.

(not reported here)[10]. Chen writes consumption growth $\dot{c}/c$ as a function of x and λ and treats the left-hand side of the equation as independent – but it is not. As evidence, Carroll et al. (1997) analyze the system formed by $\dot{c}/c$ and x rather than λ and x. One of $\dot{c}/c$ or λ is redundant in equation (6b). Intuitively, the reason is that the non-linearity affects consumption growth and habit growth proportionally. This is easy to see if we re-write the habit formation mechanism in growth form,

$$\frac{\dot{S}_t}{S_t} = B\left(\frac{c_t}{S_t}\right)^{\mu} - \delta_S. \qquad (2b')$$

On a balanced growth path, the right-hand side of (2b′) must be constant, which requires $\mu(\dot{c}_t/c_t - \dot{S}_t/S_t)$ to be zero. The non-linearity parameter μ does not affect the difference between consumption growth and habit growth and consequently cannot affect the relative trajectories of consumption and habit. The habit formation mechanism studied by Chen is insufficiently non-linear to generate multiple balanced growth paths. Proposition 1 follows directly from my direct proof that the balanced growth path is unique, in section 3.4.

3.4 A Direct Proof that the Balanced Growth Path is Unique

A manipulation of the first-order conditions proves that the balanced growth path is unique (some details are relegated to the appendix). The marginal utility of consumption is strictly positive, $U_c(c, S) > 0$, since the function $U(c, S)$ applies an infinite penalty to zero consumption. Differentiate (3a) with respect to time :

$$-\frac{\dot{U}_c}{U_c} = -\frac{\dot{\lambda}_k}{\lambda_k}\frac{\lambda_k}{U_c} + \left(\frac{\lambda_k}{U_c} - 1\right)\left(\frac{\dot{\lambda}_S}{\lambda_S} + \frac{\dot{g}_c}{g_c}\right)$$

where $\dot{U}_c$ denotes the total derivative of $U_c(c, S)$ with respect to time, and likewise for $\dot{g}_c(c, S)$. Let ϕ be a short-hand for λ_k/U_c. Using the functional form assumed for U and g, we can express $\dot{U}_c/U_c$ and $\dot{g}_c/g_c$ in terms of $\dot{c}/c$ and $\dot{S}/S$. Given $\dot{\lambda}_k/\lambda_k = \rho - r$ from equation (3b),

$$\begin{aligned}[\sigma - (\mu-1)(\phi-1)]\ \dot{c}/c\ &- [(\sigma-1)\gamma - (\mu-1)(\phi-1)]\ \dot{S}/S \\ &- (\phi-1)\ \dot{\lambda}_S/\lambda_S \ = \ (r-\rho)\phi.\end{aligned}$$

[10] In *his* footnote 9, Chen (2007) justifies the fact that his result contradicts Carroll et al. (2000) by the fact that they use different transformations of the dynamic system. But of course the method used, if correct, should not change the result. In the limit, as $\mu = 1$, the models are identical, so they must be in agreement in this special case. The fact that they are not in agreement even in this special case is immediate indication that one of them must be mistaken.

This equality must hold in and out of a balanced growth path.

We can now use balanced growth conditions to express $(\dot{S}/S)_\infty$ and $(\dot{\lambda}_S/\lambda_S)_\infty$ in terms of $(\dot{c}/c)_\infty$, where the subscript ∞ indicates that the equality must hold on a balanced growth path, but need not off of it. Any balanced growth path that satisfies the habit formation mechanism (2b) – or equivalently (2b′) – must be such that :

$$(\dot{S}/S)_\infty = (\dot{c}/c)_\infty$$

By the same token, a balanced growth path that satisfies the first-order condition (3c) must be such that :

$$(\dot{\lambda}_S/\lambda_S)_\infty = (\dot{U}_S/U_S)_\infty = -\left[(1-\sigma)\gamma + \sigma\right](\dot{c}/c)_\infty$$

Combining the balanced growth conditions with the optimality conditions,

$$\begin{aligned}(\sigma - (\mu-1)(\phi-1) - (\sigma-1)\gamma + (\mu-1)(\phi-1) + (\phi-1)\left[(1-\sigma)\gamma+\sigma\right]) \\ \times(\dot{c}/c)_\infty = (r-\rho)\phi\end{aligned}$$

Provided $\lambda_k/U_C = \phi \neq 0$, the above simplifies to :

$$\left(\frac{\dot{c}}{c}\right)_\infty = \frac{r-\rho}{\Delta} \tag{4}$$

From (4), there is a unique growth rate of consumption that satisfies both the optimality conditions and the restrictions imposed by the requirements of balanced growth. This proves part of Proposition 2. From the parameter restrictions assumed, $\sigma > 1$ and $0 < \gamma < 1$, $\Delta = (1-\sigma)\gamma+\sigma > 1$. The long-run elasticity of substitution $1/\Delta$ is independent of μ. This proves Proposition 1.

The balanced growth rate of consumption is independent of μ and is the same whether habit formation is internal or external to the agent.

4 Appendix

4.1 Appendix to Section 3.4

In this section I detail the steps involved in the differentiation of equation (3a) presented in section 3.4, following the Editor's request.

$$U_c = \lambda_k + \lambda_S g_c \tag{5}$$

$$\Rightarrow \dot{U}_c = \dot{\lambda}_k + \dot{\lambda}_S g_c + \lambda_S \dot{g}_c \tag{6}$$

$$\Rightarrow \frac{\dot{U}_c}{U_c} = \frac{\dot{\lambda}_k}{U_c} + \frac{1}{U_c}\left(\dot{\lambda}_S g_c + \lambda_S \dot{g}_c\right) \tag{7}$$

$$\Rightarrow \frac{\dot{U}_c}{U_c} = \frac{\dot{\lambda}_k}{\lambda_k}\frac{\lambda_k}{U_c} + \frac{\lambda_S g_c}{U_c}\left(\frac{\dot{\lambda}_S}{\lambda_S} + \frac{\dot{g}_c}{g_c}\right) \tag{8}$$

$$\text{From (5)} \quad \Rightarrow \lambda_S g_c = U_c - \lambda_k \tag{9}$$

$$\Rightarrow \frac{\lambda_S g_c}{U_c} = \frac{U_c - \lambda_k}{U_c} \tag{10}$$

$$\text{From (8)} - \text{(10)} \quad \Rightarrow \frac{\dot{U}_c}{U_c} = \frac{\dot{\lambda}_k}{\lambda_k}\frac{\lambda_k}{U_c} + \frac{U_c - \lambda_k}{U_c}\left(\frac{\dot{\lambda}_S}{\lambda_S} + \frac{\dot{g}_c}{g_c}\right) \tag{11}$$

$$\Rightarrow -\frac{\dot{U}_c}{U_c} = -\frac{\dot{\lambda}_k}{\lambda_k}\frac{\lambda_k}{U_c} + \left(\frac{\lambda_k}{U_c} - 1\right)\left(\frac{\dot{\lambda}_S}{\lambda_S} + \frac{\dot{g}_c}{g_c}\right) \tag{12}$$

Proof that Chen's equation (7a) and Toche's equation (7a') are equivalent.

```
> restart: alias(sigma=sig): alias(gamma=gam): alias(delta=delta_s):
```

RHS-Chen corresponds to equation (7a). RHS_toche corresponds to equation (7a').

```
> RHS_toche:=r-rho-(sig+(1-sig)*gam)*(B*x^(mu)-delta_s)-lambda*(B*mu
*x^(mu-1))*(rho+delta_s-gam*(x/lambda)-B*(1-mu+mu*gam)*x^(mu)+(sig
+(1-sig)*gam)*(B*x^(mu)-delta_s));
```

$$RHS\_toche := r - \rho - (\sigma + (1-\sigma)\gamma)(B x^{\mu} - \delta) - \lambda B \mu x^{(\mu-1)}\left(\rho + \delta - \frac{\gamma x}{\lambda} - B(1-\mu+\mu\gamma)x^{\mu} + (\sigma + (1-\sigma)\gamma)(B x^{\mu} - \delta)\right)$$

```
> RHS_chen:=r-rho+(gam*(1+(sig-1)/mu)*x/lambda-(rho+mu*delta_s+gam*(
sig-1)*delta_s)+gam*mu*B*x^(mu))*B*mu*lambda*x^(mu-1)+gam*(sig-1)*
(B^(2)*mu*lambda*x^(2*mu-1)-delta_s)-(B*x^(mu)-delta_s)*(sig+(sig+
mu-1)*B*mu*lambda*x^(mu-1));
```

$$RHS\_chen := r - \rho + \left(\frac{\gamma\left(1 + \frac{\sigma-1}{\mu}\right)x}{\lambda} - \rho - \mu\delta - \gamma(\sigma-1)\delta + \gamma\mu B x^{\mu}\right) B \mu \lambda x^{(\mu-1)} + \gamma(\sigma-1)(B^2 \mu \lambda x^{(2\mu-1)} - \delta) - (B x^{\mu} - \delta)(\sigma + (\sigma+\mu-1) B \mu \lambda x^{(\mu-1)})$$

```
> is(RHS_toche=RHS_chen);
```

true

Fig. 2. **Maple Output** Proof of the equivalence of equations (7a) and (7a'). The derivation of equations (7a) and (7a') is a tedious, error-prone task. A Maple output is provided in the interest of concision.

References

Carroll, C. D., Overland, J., Weil, D. N., December 1997. Comparison utility in a growth model. Journal of Economic Growth 2 (4), 339–67.

Carroll, C. D., Overland, J., Weil, D. N., June 2000. Saving and growth with habit formation. American Economic Review 90 (3), 341–355.

Chen, B.-L., February 2007. Multiple BGPs in a growth model with habit persistence. Journal of Money, Credit and Banking 39 (1), 25–48.

Iannaccone, L. R., 1986. Addiction and satiation. Economics Letters 21 (1), 95–99.

Ryder, H. E., Heal, G. M., January 1973. Optimum growth with intertemporally dependent preferences. Review of Economic Studies 40 (1), 1–33.